Techniques of Limits

1. Delta- Epsilon Proof:

$\lim_{x \to a} f(x) = L$ if

$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|x-a\| < d$ then $|f(x) - L| < \varepsilon$

E.g. 1 Use the definition to prove that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$$

Solution:

$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|(x,y)-(0,0)\| < d$ then

$$\left| \frac{xy^2}{x^2+y^2} - 0 \right| < \varepsilon$$

We have:

1. $0 < \|(x,y)\| < d$

2. $\left| \frac{xy^2}{x^2+y^2} \right| < \varepsilon$

$$\left| \frac{xy^2}{x^2+y^2} \right| < |x| \left(\frac{y^2}{x^2+y^2} \right)$$

$\leq |x|$ Because $\frac{y^2}{x^2+y^2} \leq 1$

$\leq |x+y|$

$= \sqrt{x^2+y^2}$

Therefore, let $d = \varepsilon$.

Proof

$0 < \| (x,y) \| < d$

$\sqrt{x^2+y^2} < d$

$= \varepsilon$

$$\begin{aligned}\sqrt{x^2+y^2} &= |x+y| \\ &\geq |x| \\ &\geq |x| \left(\frac{y^2}{x^2+y^2}\right) \\ &\geq \left|\frac{xy^2}{x^2+y^2}\right|\end{aligned}$$

$\therefore \left|\frac{xy^2}{x^2+y^2}\right| < \varepsilon$, as wanted

2. Approaching the point from various directions:

In multi-variable limits, there are an infinite number of ways to approach the point. However, if any 2 ways don't get the same result, then the limit DNE.

E.g. 2 Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

Solution:

Along the x-axis, $(x,0)$, we have:

$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$

Along the y-axis, $(0,y)$, we have:

$\lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1$ Since $1 \neq -1$, the limit DNE.

3. Plugging in the point:

If the function is continuous at the point, then we can plug the point in.

E.g. 3 Calculate $\lim_{(x,y)\to(1,2)} \frac{(x+y)^2}{x^2+y^2}$

Solution:

Since the function is cont at (1,2), we can plug it in.

$$\lim_{(x,y)\to(1,2)} \frac{(x+y)^2}{x^2+y^2}$$

$$= \frac{9}{5}$$

4. Substitution:

Recall that $r = \sqrt{x^2+y^2}$. Therefore, given x^2+y^2, we can sub r for it.

E.g. 4 Evaluate $\lim_{(x,y)\to(0,0)} (x^2+y^2+1)^{\left(\frac{1}{x^2+y^2}\right)}$

Solution:

Let $r^2 = x^2+y^2$

$(x,y)\to(0,0)$ means $r\to 0$

$$\lim_{r\to 0} (r^2+1)^{\left(\frac{1}{r^2}\right)}$$

$$P = \lim_{r\to 0} (r^2+1)^{\left(\frac{1}{r^2}\right)}$$

$$\ln(P) = \ln\left(\lim_{r\to 0} (r^2+1)^{\left(\frac{1}{r^2}\right)}\right)$$

$$\ln(p) = \lim_{r\to 0}\left(\ln\left((r^2+1)^{\frac{1}{r^2}}\right)\right)$$

$$= \lim_{r\to 0}\left(\frac{\ln(r^2+1)}{r^2}\right)$$

$$= \lim_{r\to 0}\frac{\frac{2r}{r^2+1}}{2r} \quad \text{(L'hopital's Rule)}$$

$$= \lim_{r\to 0}\frac{1}{r^2+1}$$

$$=1$$

$$\ln(P) = 1$$

$$P = e$$

$$\therefore \lim_{(x,y)\to(0,0)} (x^2+y^2+1)^{\left(\frac{1}{x^2+y^2}\right)} = e$$

5. Squeeze Theorem:

E.g. 5 Evaluate $\lim_{(x,y)\to(0,0)} (x)(y)\left(\sin\left(\frac{1}{x+y}\right)\right)$

Solution:

$$-1 \le \sin\left(\frac{1}{x+y}\right) \le 1$$

Suppose $xy > 0$

$$\lim_{(x,y)\to(0,0)} -(xy) \le \lim_{(x,y)\to(0,0)} (xy)\left(\sin\left(\frac{1}{x+y}\right)\right) \le \lim_{(x,y)\to(0,0)} xy$$

$$0 \le \lim_{(x,y)\to(0,0)} (xy)\left(\sin\left(\frac{1}{x+y}\right)\right) \le 0$$

Suppose $xy < 0$

$$\lim_{(x,y)\to(0,0)} -(xy) = 0$$

$$\lim_{(x,y)\to(0,0)} (xy) = 0$$

$$\lim_{(x,y)\to(0,0)} -(xy) \leq \lim_{(x,y)\to(0,0)} (xy)(\sin(\tfrac{1}{x+y})) \leq \lim_{(x,y)\to(0,0)} xy$$

$$0 \leq \lim_{(x,y)\to(0,0)} (xy)(\sin(\tfrac{1}{x+y})) \leq 0$$

$\therefore$ By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} (xy)(\sin(\frac{1}{x+y})) = 0$

6. Taylor Series:

Here are some taylor series:

1. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

2. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

3. $\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$

E.g. 6 Evaluate $\lim_{(x,y)\to(0,0)} \frac{\ln(xy)}{xy-1}$

Solution:

$$\ln(xy) = (xy-1) - \frac{1}{2}(xy-1)^2 + \frac{1}{3}(xy-1)^3 - \ldots$$

$$\lim_{(x,y)\to(0,0)} \frac{(xy-1) - \frac{1}{2}(xy-1)^2 + \frac{1}{3}(xy-1)^3 - \ldots}{xy-1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{(xy-1)}\left[1 - \frac{1}{2}(xy-1) + \frac{1}{3}(xy-1)^2 - \ldots\right]}{\cancel{xy-1}}$$

$$= 1$$

7. Using the property $\lim_{x\to c} f(x) = L$ iff $\lim_{x\to c} |f(x)-L| = 0$

E.g. 7. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Solution:

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{|x|\,|y|}{\sqrt{x^2+y^2}}$$

$$\le \frac{|x+y|\,|x+y|}{\sqrt{x^2+y^2}}$$

$$\le \frac{\cancel{\sqrt{x^2+y^2}}\,\sqrt{x^2+y^2}}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \sqrt{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \ge 0 \quad \therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

8. Basic Limits and Inequalities:

1. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin(x)}$

2. $\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$

3. $\lim_{x \to 0} \frac{\cos(x)}{x} =$ DNE

4. $\lim_{x \to \infty} \frac{\cos(x)}{x} = 0 = \lim_{x \to 0} \frac{x}{\cos(x)}$

5. $|x| \leq |x+y|$

6. $|x+y| \leq |x| + |y|$

7. $|x-a| \leq |(x-a)+(y-b)|$

8. $|xy| \leq (|x|)(|y|)$